ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, Nos. 1–2, 1985

CONVEX HULLS OF GENERALIZED MOMENT CURVES

BY
ZEEV SMILANSKY
Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT
We consider the family of curves in $\mathbf{R}^4$:

$$M_{pq} = \{(\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt) : t \in \mathbf{R}\},$$

where p and q are positive integers, and determine the facial structure of the convex hull of these curves.

1. Introduction

In this paper we determine the facial structure of the convex hull of trigonometric curves of the form:

$$M_{pq} = \{(\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt) : 0 \leqq t < 1\} \subseteq \mathbf{R}^4,$$

where p and q are positive integers. In a sequel to this paper we shall consider the convex hull of v ($v \geqq 5$) evenly spaced points on M_{pq}.

Interest in these curves arises from the following considerations. Firstly, these curves are a natural generalization of the "classical" 4-dimensional (trigonometric) moment curve

$$M_{12} = (\cos 2\pi t, \sin 2\pi t, \cos 4\pi t, \sin 4\pi t).$$

Secondly, they exhibit a high degree of symmetry (see §2 below). Lastly, the structure of their convex hull can be completely determined, which in itself is an interesting point. In fact, the method by which we shall obtain the structure of M_{pq} can be thought of as a generalization of the methods that Gale used to determine the structure of the cyclic polytopes (cf. Gale [1]). In the same way that Gale studied properties of polynomial functions derived from the moment

Received November 23, 1984

curve, we shall study trigonometric functions related to M_{pq}, to obtain our results.

2. Basic properties of C_{pq}

DEFINITION. For positive integers p, q and for $t \in \mathbf{R}$, define:

(i) $z_{pq}(t) = (\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt)$,

(ii) $M_{pq} = \{z_{pq}(t) : 0 \leqq t < 1\}$,

(iii) $C_{pq} = \operatorname{conv} M_{pq}$.

The following assertions are obvious:

(a) $z_{pq}(t) = z_{pq}(t+1)$.

(b) $\| z_{pq}(t) \| = \sqrt{2}$.

(c) If $(p,q) = 1$ (i.e., if p and q are relatively prime) then M_{pq} is a simple closed curve.

Let $d = (p,q)$. It is immediate that $C_{pq} = C_{p/d,q/d}$, and therefore we shall assume from now on that $(p,q) = 1$. We shall also assume that $1 \leqq p < q$. Define the following orthogonal matrices:

$$A(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix},$$

$$T_{pq}(\alpha) = \begin{pmatrix} A(p\alpha) & 0 \\ 0 & A(q\alpha) \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

With these definitions, the following relations hold:

$$z_{pq}(t) \cdot T_{pq}(\alpha) = z_{pq}(t+\alpha),$$

$$z_{pq}(t) \cdot R = z_{pq}(-t).$$

The matrices $T_{pq}(\alpha)$, $0 \leqq \alpha < 1$, together with R, generate a group D_{pq} of symmetries of M_{pq}, and thus of C_{pq}. Every member of D_{pq} induces a combinatorial automorphism of C_{pq}. Moreover, D_{pq} acts transitively on M_{pq}, and thus, in order to study the combinatorial structure of C_{pq}, i.e., its faces and facelets,[†] it

[†]If $K \subset \mathbf{R}^d$ is convex and $S \subset K$, then S is a *facelet* of K provided there is a flat L of $\mathbf{R}^d$ such that $S = L \cap K$ and $K \backslash L$ is convex; Grünbaum [2] uses the term "poonem".

suffices to consider those faces and facelets which contain a given vertex, say $z_{pq}(0) = (1,0,1,0)$. Every other face and facelet is obtained from these by the action of some member of D_{pq}, i.e., by some rotation or reflection. (Actually, the rotations are already transitive on M_{pq}.)

3. Statement of the results

3.1. For the rest of the paper we fix p and q and write C for C_{pq}, M for M_{pq} and z for z_{pq}.

NOTATION. Given relatively prime integers p and q, $1 \leqq p < q$, there is a unique pair of integers k and l, $0 \leqq k < p$, $1 \leqq l < q$, such that $lp - kq = 1$. We denote by J_{pq} the open interval $]k/p, l/q[\subset]0,1[$. We also define:

$$J^*_{pq} = J_{pq} \cup (1 - J_{pq}) = \left]\frac{k}{p}, \frac{l}{q}\right[\cup \left]\frac{q-l}{q}, \frac{p-k}{p}\right[.$$

REMARKS. J_{pq} and $1 - J_{pq}$ are the only open intervals in $[0,1[$ of length $1/pq$, having endpoints of the form i/p, j/q. (The length of any interval with such endpoints is always a multiple of $1/pq$.) Note that $\mathrm{cl}\, J_{pq}$ and $\mathrm{cl}(1 - J_{pq})$ are disjoint closed intervals (even when considered as intervals in $\mathbf{R}/\mathbf{Z}$), whenever $p \geqq 3$.

The following theorem is the main result of this paper:

THEOREM 1. (a) *The proper faces of C are*:

(1) $\{z(t)\}$, $t \in [0,1[$. *These are* 0-*faces* (*i.e.*, 0-*dimensional faces*).

(2) $[z(s), z(t)]$, $0 \leqq s < t < 1$, $t - s \in J^*_{pq}$. *These are* 1-*faces.*

(3) $\mathrm{conv}\{z(t + i/p),\ i = 0,1,\ldots,p-1\}$, $0 \leqq t < 1/p$. *If $p \geqq 3$ these are* 2-*faces which are regular p-gons. If $p = 2$, these are* 1-*faces that were not listed in* (2). *If $p = 1$, these are* 0-*faces that were* (*obviously*) *listed in* (1).

(4) $\mathrm{conv}\{z(t + i/q),\ i = 0,1,\ldots,q-1\}$, $0 \leqq t < 1/q$. *Same comments as in* (3), *according to whether $q \geqq 3$ or $q = 2$.*

(5) *There are no* 3-*faces.*

(b) *The facelets of C that are not faces are the edges of the p-gons listed in* (a)(3) (*when $p \geqq 3$*) *and of the q-gons listed in* (a)(4) (*when $q \geqq 3$*).

3.2. REMARKS. The most interesting (and difficult) part of Theorem 1 is statement (a)(2), which characterizes those pairs of points of M which span edges of C. We dispense with the rest of the faces of C — vertices and 2-dimensional faces (which can degenerate into edges in some cases) — in the following remarks:

(1) The 0-faces of C are those listed in (a)(1), since $C = \operatorname{conv} M$ and $M \subseteq \operatorname{bd} B(0, \sqrt{2})$. Thus every point of M is an exposed point of C.

(2) For statement (a)(3), consider "special" support hyperplanes H_v of C of the form $H_v = \{u \in \mathbf{R}^4 : \langle u, v \rangle = \delta\}$, where $\delta \geqq 0$ and $v = (v_1, v_2, 0, 0)$ is a unit vector. Writing v as $(\cos 2\pi p t_0, \sin 2\pi p t_0, 0, 0)$, we obtain: $\langle z(t), v \rangle = \cos 2\pi p(t - t_0)$. Since H_v supports C, δ must be 1, and therefore

$$H_v \cap C = \operatorname{conv}\{z(t) : \cos 2\pi p(t - t_0) = 1\} = \operatorname{conv}\{z(t_0 + i/p), i = 0, 1, \ldots, p-1\}.$$

Thus (a)(3) lists exactly all the faces of C supported by these "special" hyperplanes. A simple computation (see Remark (3) below) shows that these faces are 2-dimensional regular polygons (if $p \geqq 3$).

For (a)(4), the same considerations hold. These faces are supported by hyperplanes H_v, where $v = (0, 0, \cos 2\pi q t_0, \sin 2\pi q t_0)$.

(3) As to part (b) of Theorem 1, note that every proper facelet of a convex set K is a facelet of a proper face of K (see Grünbaum [2], p. 27). Assuming part (a) of the theorem, the only candidates for facelets that are not faces are the edges of the polygons in (a)(3) and (a)(4). These are indeed facelets, being faces of faces. Consider such a polygon, say

$$F = \operatorname{conv}\left\{ z(0), z\left(\frac{1}{p}\right), \ldots, z\left(\frac{p-1}{p}\right)\right\}.$$

Two vertices $z(i/p)$ and $z(j/p)$ of F are adjacent (determine an edge) iff qi and qj are "adjacent" modulo p, i.e., if

$$q \cdot (i - j) \equiv \pm 1 (\operatorname{mod} p)$$

or

$$i - j \equiv \pm k (\operatorname{mod} p),$$

where $lp - kq = 1$ (see notation in 3.1). We conclude that the edges of F are of the form $[z(t), z(t + k/p)]$ and $k/p \in \operatorname{bd} J^*_{pq}$. ($0 < k < p$, by definition and since $p \geqq 3 > 1$.) Similarly, the edges of the q-gons in (a)(4) are of the form $[z(s), z(s + l/q)]$, $l/q \in \operatorname{bd} J^*_{pq}$. These facelets of C are limits in the Hausdorff metric of the edges listed in (a)(2). Assuming part (a), these facelets are not faces, since k/p and l/q are not in J^*_{pq}.

In view of these remarks, what remains to be proven of Theorem 1 can be summarized in:

THEOREM 2. (a) *$[z(s), z(t)]$ is a face of C whenever $0 \leqq s < t < 1$ and $t - s \in J^*_{pq}$.*

(b) *If F is a proper face of C,* $\dim F > 0$, *and if F lies in a supporting hyperplane* H_v, *where* $v = (v_1, v_2, v_3, v_4)$ *and neither* $v_1 = v_2 = 0$ *nor* $v_3 = v_4 = 0$ *hold, then:* $F = [z(s), z(t)]$, *for some* $0 \leqq s < t < 1$, *such that* $t - s \in J^*_{pq}$.

3.3. We illustrate Theorem 1 with a few examples.

(i) The case $p = 1$, $q = 2$. M is now the classical trigonometric moment curve, and we expect to find that any two distinct points on M determine a 1-face. Indeed, $J^*_{pq} =]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$, so that $[z(s), z(t)]$ is a 1-face whenever $0 \leqq s < t < 1$ and $t - s \neq \frac{1}{2}$. Also, since $p = 1$ and $q = 2$, there are no 2-faces, and the q-gons listed in part (a)(4) of Theorem 1 are the intervals $[z(s), z(s + \frac{1}{2})]$, $0 \leqq s < \frac{1}{2}$ as expected.

(ii) $p = 1$, $q \geqq 3$. In this case $J^*_{pq} =]0, 1/q[\cup]1 - 1/q, 1[$. The 1-faces are $[z(s), z(t)]$, where $0 \leqq s < t < 1$, and $t - s \in J^*_{pq}$. The 2-faces are q-gons, and $[z(t), z(t + 1/q)]$ is a facelet that is not a face, for every $t \in [0, 1[$.

(iii) $p = 2$, $q \geqq 3$. Here $J^*_{pq} =]\frac{1}{2} - 1/2q, \frac{1}{2}[\cup]\frac{1}{2}, \frac{1}{2} + 1/2q[$. Note that in this case $[z(t), z(t + \frac{1}{2})]$ is a face, and not only a facelet.

3.4. Remark. One of the most interesting properties of the classical moment curve is its neighbourliness, i.e., the fact that every two distinct points on $M_{1,2}$ determine an edge of $C_{1,2}$. Our results show that for a given point z^* of M_{pq}, the set of points $\{t \in [0, 1[: [z^*, z(t)]$ is an edge of $C_{pq}\}$ is a union of two intervals with length $1/pq$. In this sense one could say that C_{pq} is "$2/pq$ neighbourly". Note also that the position of these intervals depends on number theoretic properties of p and q.

4. Reduction of the problem

We begin now to tackle the problem of determining the edges of C, a problem which will occupy us for the rest of this paper. Suppose that F is a face of C. $C = \operatorname{conv} M$, and thus $F = \operatorname{conv} A$ for some $A \subset M$. Note that 0 is the barycenter of M; therefore, if H is a supporting hyperplane and $F = C \cap H$ is a proper face of C, then $0 \notin H$, and $C \subseteq H^-$. (H^- is the closed half space bounded by H that contains 0, while H^+ is the one that does not.) Thus we can restate the problem as follows: What are the subsets A of M, such that for some hyperplane $H \subset \mathbf{R}^4 \setminus \{0\}$, $M \cap H^+ = M \cap H = A$.

Definitions.

(1) Define $\eta : \mathbf{R}^2 \to \mathbf{R}^4$ by:

$$\eta(\theta, \varphi) = (\cos 2\pi\theta, \sin 2\pi\theta, \cos 2\pi\varphi, \sin 2\pi\varphi).$$

(2) $\Sigma = \eta^{-1}(M) \subseteq \mathbf{R}^2$.

(3) For a hyperplane $H_v = \{u \in \mathbf{R}^4 : \langle u, v \rangle = 1\}$, define:

$S(H_v) = \eta^{-1}(H_v^+) = \{(\theta, \varphi) \in \mathbf{R}^2 : \langle (\cos 2\pi\theta, \sin 2\pi\theta, \cos 2\pi\varphi, \sin 2\pi\varphi), v \rangle \geqq 1\}$.

Let us consider some of the properties of η, Σ and $S(H_v)$. Note first that η maps $\mathbf{R}^2$ onto the torus $T^2 = \{(\cos\alpha, \sin\alpha, \cos\beta, \sin\beta) : (\alpha, \beta) \in \mathbf{R}^2\}$. (Evidently $M \subseteq T^2 \subset \operatorname{bd} B(0, \sqrt{2})$.) Note also that for every $x \in T^2$, $\eta^{-1}(x) = \eta^{-1}(x) + \mathbf{Z}^2$, where $\mathbf{Z}^2 = \{(n, m) : n, m \in \mathbf{Z}\}$ is the standard unit lattice. Therefore, every η-preimage, and in particular Σ and $S(H_v)$, is doubly periodic, with periods (1,0) and (0,1). Accordingly, we shall often confine our attention to the fundamental square

$$E = [-\tfrac{1}{2}, \tfrac{1}{2}] \times [-\tfrac{1}{2}, \tfrac{1}{2}].$$

The planar set Σ is easily described. Define a line $L_0 = \{(pt, qt),\ t \in \mathbf{R}\}$, and note that since $\eta(pt, qt) = z(t)$, $M = \eta(L_0)$ and thus $\Sigma = \eta^{-1}(M) = L_0 + \mathbf{Z}^2$. Another important observation is that for any t_0,

$$\eta^{-1}(z(t_0)) = (pt_0, qt_0) + \mathbf{Z}^2. \tag{4.1}$$

A simple computation shows that $\Sigma = L_0 + \mathbf{Z} \cdot (1/q, 0) = L_0 + \mathbf{Z} \cdot (0, 1/p)$, i.e., Σ is the union of a sequence of lines having slope q/p. The horizontal distance between two adjacent lines is $1/q$, and the vertical distance is $1/p$.

Consider the set $S(H_v)$, defined in (3) above. Writing the vector $v = (v_1, v_2, v_3, v_4)$ as

$$v = (a\cos 2\pi\theta_0, a\sin 2\pi\theta_0, b\cos 2\pi\varphi_0, b\sin 2\pi\varphi_0), \tag{4.2}$$

where $a = \sqrt{v_1^2 + v_2^2} \geqq 0$ and $b = \sqrt{v_3^2 + v_4^2} \geqq 0$, we obtain

$$S(H_v) = \{(\theta, \varphi) \in \mathbf{R}^2 : a\cos 2\pi(\theta - \theta_0) + b\cos 2\pi(\varphi - \varphi_0) \geqq 1\}. \tag{4.3}$$

Thus, we can write $S(H_v) = S(a, b, \theta_0, \varphi_0)$, for the suitable quadruple a, b, θ_0, φ_0. Note that if H_v is a support hyperplane of M, and $S(H_v) = S(a, b, \theta_0, \varphi_0)$, then $a + b \geqq 1$; for otherwise $S(H_v) = \varnothing$, i.e., $H_v \cap T^2 = \varnothing$. Conversely, every real a, b, θ_0, φ_0, with a, b nonnegative, $a + b \geqq 1$, determines a hyperplane H_v, where v is defined as in (4.2), and $H_v \cap T^2 \neq \varnothing$. Also, (4.3) implies that

$$\begin{aligned} \operatorname{bd} S(H_v) &= \operatorname{bd} S(a, b, \theta_0, \varphi_0) \\ &= \{(\theta, \varphi) \in \mathbf{R}^2 : a\cos 2\pi(\theta - \theta_0) + b\cos 2\pi(\varphi - \varphi_0) = 1\}. \end{aligned}$$

The following lemma further describes $S(H_v)$.

LEMMA 3. *Let* $a \geqq 0$, $b \geqq 0$, $S = S(a,b,0,0)$. *Then* S *has the following properties*:

(a) S *is symmetric with respect to reflection in each of the coordinate axes, and therefore centrally symmetric.*

(b) *If* $\operatorname{int} S \neq \emptyset$ (*i.e.*, if $a+b>1$), *then* $E \cap S$ *is strongly starshaped with respect to the origin, in the sense that* $u \in E \cap S$ *and* $0 \leqq \lambda < 1$ *imply* $\lambda u \in \operatorname{int}(E \cap S)$. *Moreover, if* $u = (\theta, \varphi)$ *then the entire rectangle with vertices* $(\pm\theta, \pm\varphi)$ *lies in* $E \cap S$.

(c) *If* $a>0$, $b>0$ *and if* $\operatorname{int} S \neq \emptyset$, *then a line tangent to* $\operatorname{bd} S$ *at a point* (θ_0, φ_0) *in* $\operatorname{int} E$ *has positive slope if* $\theta_0 \cdot \varphi_0 < 0$, *negative slope if* $\theta_0 \cdot \varphi_0 > 0$, *zero slope if* $\theta_0 = 0$ *and infinite slope if* $\varphi_0 = 0$.

PROOF. (a) and (b) follow immediately from the definition of S. (c) follows by a simple computation of the derivative of the implicit function $\varphi(\theta)$ (or $\theta(\varphi)$) defined by $a\cos 2\pi\theta + b\cos 2\pi\varphi = 1$. ∎

REMARK. In fact, $S(H_v)$ will be a convex planar set in the cases that we shall consider. However, we shall not prove this, but rather use weaker properties of $S(H_v)$. It might be helpful to visualize $S(H_v)$ as looking rather like an ellipse (although it is not), with its major axes parallel to the axes of the plane.

We are now able to reformulate the 4-dimensional problem, stated in the beginning of this section, in two dimensions, translating the sets H^+, H, M and A into $\eta^{-1}(H^+) = S(H)$, $\eta^{-1}(H) = \operatorname{bd} S(H)$, $\eta^{-1}(M) = \Sigma$ and $\eta^{-1}(A) = B$: What are the subsets B of Σ, such that there exist real numbers a, b ($a \geqq 0$, $b \geqq 0$) and θ_0, φ_0, and for which

$$\Sigma \cap S(a,b,\theta_0,\varphi_0) = \Sigma \cap \operatorname{bd} S(a,b,\theta_0,\varphi_0) = B.$$

Ann illustration of Σ (for the case $p=3$ and $q=5$) and $S(H_v)$ is given in Fig. 1.

LEMMA 4. *Suppose that* $0 \leqq s < t < 1$. *Then* $t - s \in J^*_{pq}$ *if and only if* $\eta^{-1}(z(s)) = u + \mathbf{Z}^2$ *and* $\eta^{-1}(z(t)) = v + \mathbf{Z}^2$, *where* u *and* v *are points that lie on adjacent lines of* Σ *and determine a line with a negative slope.*

PROOF. By (4.1), replacement of s and t by $s+\alpha$ and $t+\alpha$ will translate u and v by the constant vector $(p\alpha, q\alpha) \in L_0$ (see description of Σ above). This translation maps each line of Σ onto itself. Accordingly, it suffices to consider the case $s=0$. Also, we can clearly replace u and v by $u+(n,m)$ and $v+(n,m)$, where $(n,m) \in \mathbf{Z}^2$. Thus w.l.o.g. we take $s=0$ and $u=(0,0)$. ($\eta^{-1}(z(0)) = (0,0) + \mathbf{Z}^2$.)

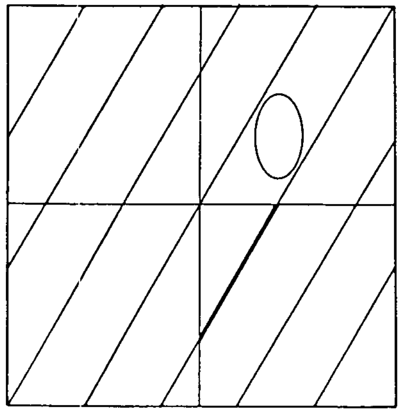

Fig. 1. $\Sigma = \Sigma_{3,5}$, $S = S(0.8,0.25,0.25,0.25)$; I is heavily marked.

Define $I = (L_0 + (1/q,0)) \cap Q_4$, where Q_4 is the interior of the 4th quadrant. Thus $I =](0,-1/p),(1/q,0)[$ (see Fig. 1). Note that the set of points of Σ that lie on a line adjacent to L_0 and together with (0,0) determine a line of negative slope is precisely $I \cup -I$; thus we have to show that $t \in J^*_{pq} \Leftrightarrow \eta^{-1}(z(t)) \subset (I \cup -I) + \mathbf{Z}^2$.

Recall that $J_{pq} =]k/p, l/q[$, where $lp - kq = 1$, and consider the mapping $t \to (pt, qt) - (k,l)$ $(t \in [0,1[)$. This mapping is one-to-one between J_{pq} and I: it is an affine mapping, and we have

$$\frac{k}{p} \to \left(k,\frac{kq}{p}\right) - (k,l) = \left(0,\frac{kq-lp}{p}\right) = \left(0,-\frac{1}{p}\right)$$

and

$$\frac{l}{q} \to \left(\frac{lp}{q},l\right) - (k,l) = \left(\frac{lp-kq}{q},0\right) = \left(\frac{1}{q},0\right).$$

Thus, $t \in J_{pq} \Rightarrow \eta^{-1}(z(t)) = (pt,qt) + \mathbf{Z}^2 = (pt,qt) - (k,l) + \mathbf{Z}^2 \subset I + \mathbf{Z}^2$.

Conversely, if $\eta^{-1}(z(t)) \subset I + \mathbf{Z}^2$, then

$$\eta^{-1}(z(t)) = (pt^*,qt^*) - (k,l) + \mathbf{Z}^2 = (pt^*,qt^*) + \mathbf{Z}^2$$

for some $t^* \in J_{pq}$. But $\eta^{-1}(z(t)) = (pt,qt) + \mathbf{Z}^2$. Since p and q are relatively prime, it follows that $t \equiv t^* \pmod 1$, and since both t and t^* belong to $[0,1[$, $t = t^* \in J_{pq}$.

Similarly, $t \in 1 - J_{pq}$ iff $\eta^{-1}(z(t)) \subset (-I) + \mathbf{Z}^2$. Thus the proof is complete. ■

We can finally state a "translated" version of Theorem 2. This version is the one we shall prove.

DEFINITION. Let $S = S(a, b, \theta_0, \varphi_0)$, and $D \subseteq \Sigma$. We shall say that S supports Σ at D if $\Sigma \cap S = \Sigma \cap \operatorname{bd} S = D$. (Obviously S supports Σ at D if and only if the corresponding hyperplane supports C at $\operatorname{conv}\{\eta(x) : x \in D\}$.)

THEOREM 5.

(a) *If u and v lie on adjacent lines of Σ and determine a line of negative slope, then there are a, b, θ_0, φ_0, $a > 0$, $b > 0$, such that $S(a, b, \theta_0, \varphi_0)$ supports Σ at $\{u, v\} + \mathbf{Z}^2$ (i.e., u and v correspond to the endpoints of an edge of C_{pq}).*

(b) *Suppose that $a > 0$ and $b > 0$. If $S = S(a, b, \theta_0, \varphi_0)$ is such that S supports Σ at D, and $|D/\mathbf{Z}^2| \geqq 2$, then $D = \{u, v\} + \mathbf{Z}^2$, where u and v lie on adjacent lines of Σ and determine a line of negative slope (i.e., every pair of endpoints of an edge of C_{pq} is obtained in this way).*

The preceding discussion shows that Theorem 5 implies Theorem 2.

The following tedious lemma is, unfortunately, crucial to the proof of Theorem 5. It proves a unimodality property of a certain class of trigonometric polynomials.

DEFINITION. Assume that $c < d$ and that $I \subseteq R$ is any interval with endpoints c and d.

(1) A differentiable function $\varphi : I \to R$ is *strictly unimodal on I* if for some $t_0 \in [c, d]$, $\varphi'(t) > 0$ for all $c < t < t_0$ and $\varphi'(t) < 0$ for all $t_0 < t < d$. t_0 is called the *peak* of φ. (Note that t_0 may be an endpoint of I.)

(2) If $g : \mathbf{R}^2 \to \mathbf{R}$ is differentiable and $J = \lambda(I)$ is a line segment in the plane, where $\lambda : \mathbf{R} \to \mathbf{R}^2$ is an affine function, then g is *strictly unimodal on J* if the function $\varphi = g \circ \lambda$ is strictly unimodal on I.

LEMMA 6. *Let $E = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbf{R}^2$, and let A be a line in $\mathbf{R}^2$ that does not coincide with a coordinate axis. Assume that A intersects both coordinate axes in E, and at least one of them in* $\operatorname{int} E$. *Define*: $f(x, y) = a \cos 2\pi x + b \cos 2\pi y$, *where* $a \geqq 0$, $b \geqq 0$, $a + b > 0$. Then f *is strictly unimodal on* $A \cap E$.

PROOF. The lemma holds trivially if $a = 0$ or $b = 0$, or if $(0,0) \in A$, since in these cases f is the sum of at most two strictly unimodal functions with the same peak. Assume therefore that $a > 0$, $b > 0$ and $(0,0) \notin A$. Using the symmetries of f and E we can reduce the problem to the following case: A cuts the x-axis at

$(x_0, 0)$ and the y-axis at $(0, y_0)$, where $-\frac{1}{2} < x_0 < 0$, $0 < -x_0 \leqq y_0 \leqq \frac{1}{2}$. Let (u_x, u_y) and (v_x, v_y) be the endpoints of $A \cap E$, and assume that the points (u_x, u_y), $(x_0, 0)$, $(0, y_0)$, (v_x, v_y) appear in this order on $A \cap E$. ((v_x, v_y) may coincide with $(0, y_0)$; see Fig. 2.) Under our assumptions, the equation of A is $y = \alpha x + \beta$, where $\alpha \geqq 1$, $0 < \beta \leqq \frac{1}{2}$, and $\beta/\alpha = -x_0 < \frac{1}{2}$. We must show that $\varphi(x) = a\cos 2\pi x + b\cos 2\pi(\alpha x + \beta)$ is strictly unimodal on $[u_x, v_x]$. Define $I =]-\beta/\alpha, 0[$ and note that $\varphi'(x) > 0$ for $x \in]u_x, -\beta/\alpha]$, and $\varphi'(x) < 0$ for $x \in [0, v_x[$. (The last statement is trivial if $v_x > 0$, and void if $v_x = 0$.) Thus it suffices to show that φ is unimodal on I.

We consider three cases:

(i) $\alpha = 1$. In this case $\varphi(x) = a\cos 2\pi x + b\cos 2\pi(x + \beta)$ can be written as $a\cos 2\pi(x + \gamma)$ and its period is equal to 1. Also $v_x > 0$, φ increases to the left of I, decreases to the right of I, $|I| < \frac{1}{2}$, and thus φ must be strictly unimodal on I.

(ii) $\alpha > 1$, $\varphi'(0) = 0$ (and therefore $\beta = \frac{1}{2}$), and $\varphi''(0) = 4\pi^2(b\alpha^2 - a) \leqq 0$. We shall show that in this case $\varphi'(x) > 0$ for all $x \in I$. We have

$$\varphi'(x) = 2\pi(-a\sin 2\pi x + b\alpha\sin 2\pi\alpha x),$$

$$\varphi''(x) = 4\pi^2(-a\cos 2\pi x + b\alpha^2\cos 2\pi\alpha x).$$

Note that $-\frac{1}{2} < \alpha x < x < 0$ for all $x \in I$.

For $-\frac{1}{2} \leqq x \leqq -\frac{1}{4}$, we have:

$$\varphi'(x) > 2\pi(b\alpha\sin 2\pi x - a\sin 2\pi x) \geqq 2\pi(b\alpha^2 - a)\sin 2\pi x \geqq 0.$$

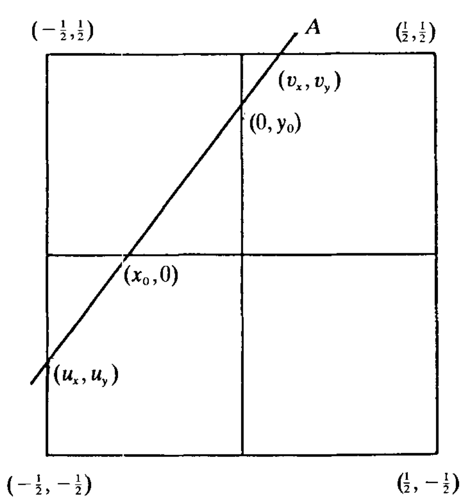

Fig. 2.

For $-\frac{1}{4} < x < 0$, we shall show that $\varphi''(x) < 0$; thus $\varphi'(x)$ is decreasing, and $\varphi'(x) > \varphi'(0) = 0$. Indeed, in this interval $\cos 2\pi x > 0$. If $\cos 2\pi \alpha x \leqq 0$ then $\varphi''(x)$ is clearly negative, and otherwise,

$$\varphi''(x) \leqq 4\pi^2(-a\cos 2\pi x + a\cos 2\pi\alpha x) < 0.$$

(iii) $\alpha > 1$, $\varphi'(0) < 0$ (or $\varphi'(0) = 0$ and $\varphi''(0) > 0$). Thus $\varphi'(x) < 0$ for all $x \in I$ sufficiently close to 0. Recall that $\varphi'(-\beta/\alpha) > 0$. Thus it suffices to show that $\varphi'(x) = 0$ for at most one x in I.

$$\varphi'(x) = -2\pi(a\sin 2\pi x + b\alpha \sin 2\pi(\alpha x + \beta)),$$

$\sin 2\pi x < 0$ and $\sin 2\pi(\alpha x + \beta) > 0$ for all $x \in I$, and therefore $\varphi'(x) = 0$ if and only if

$$g(x) = \frac{\sin 2\pi(\alpha x + \beta)}{\sin 2\pi x} = -\frac{a}{b\alpha}.$$

We shall show that $g'(x) \neq 0$ for all $x \in I$, and thus $g(x)$ cannot attain the same value twice in I.

$$g'(x) = 0 \Leftrightarrow \alpha \cos 2\pi(\alpha x + \beta)\sin 2\pi x - \sin 2\pi(\alpha x + \beta)\cos 2\pi x = 0,$$

$$\Leftrightarrow \alpha \operatorname{ctg} 2\pi(\alpha x + \beta) - \operatorname{ctg} 2\pi x = 0.$$

Finally, we show that $h(x) = \alpha \operatorname{ctg} 2\pi(\alpha x + \beta) - \operatorname{ctg} 2\pi x$ is nonzero for all $x \in I$. Note that $\alpha \operatorname{ctg} 2\pi(\alpha x + \beta)$ and $\operatorname{ctg} 2\pi x$ are both decreasing on I. Assume, on the contrary, that $x_1 \in I$ and $h(x_1) = 0$. Put

$$r = \alpha \operatorname{ctg} 2\pi(\alpha x_1 + \beta) = \operatorname{ctg} 2\pi x_1,$$

$$\begin{aligned} h'(x_1) &= (\alpha \operatorname{ctg} 2\pi(\alpha x + \beta) - \operatorname{ctg} 2\pi x)'_{x=x_1} \\ &= 2\pi(-\alpha^2(1 + \operatorname{ctg}^2 2\pi(\alpha x_1 + \beta)) + (1 + \operatorname{ctg}^2(2\pi x_1))) \\ &= 2\pi(-\alpha^2 - r^2 + 1 + r^2) = 2\pi(1 - \alpha^2) < 0, \end{aligned}$$

thus $h'(x_1) < 0$ whenever $h(x_1) = 0$, and therefore $h(x_1) = 0$ implies $h(x) < 0$ for all $x_1 < x < 0$. But a routine check shows that $h(x)$ is nonnegative for negative x, near zero. In fact, $\lim_{x \uparrow 0} h(x) = \infty$ if $\beta \neq \frac{1}{2}$; if $\beta = \frac{1}{2}$, $\lim_{x \uparrow 0} h(x) = 0$ and $h(x) > 0$ for negative x sufficiently near zero. (This can be verified, e.g., by checking the Taylor series expansion of cotangent x.) Thus $h(x) > 0$ for all $x \in I$. ■

COROLLARY 7. *Let a, b, A be as in Lemma* 6, *and* $S = S(a, b, 0, 0)$ *then*:
(a) $|E \cap A \cap \operatorname{bd} S| \leqq 2$.
(b) *If* $\{E \cap A \cap \operatorname{bd} S\} = \{u, v\}$ *then* $]u, v[\subseteq \operatorname{int} S$ *and* $E \cap A \cap S = [u, v]$.

(c) *If $a > 0$ and $b > 0$, and if the line A is tangent to the curve* $\mathrm{bd}\, S$ *at a point* $v \in \mathrm{int}\, E$, *then* $E \cap A \cap S = E \cap A \cap \mathrm{bd}\, S = \{v\}$.

PROOF. See the definition of S, Formula 4.3, and use Lemma 6. (Recall that $\mathrm{bd}\, S$ is the level set $\{(\theta,\varphi) \in \mathbf{R}^2 : f(\theta,\varphi) = 1\}$ of the function $f(\theta,\varphi) = a \cos 2\pi\theta + b \cos 2\pi\varphi$ and that f is strictly unimodal on $E \cap A$.) ∎

LEMMA 8. *Let Σ' be a translate of Σ, and let A be a line of Σ' nearest to the origin. Then A cuts the y-axis in E and the x-axis in* $\mathrm{int}\, E$. (*Thus Lemma* 6 *and Corollary* 7 *apply to A.*)

PROOF. Immediate, recalling that the horizontal distance between adjacent lines of Σ is $1/q \leqq \frac{1}{2}$, and the vertical distance is $1/p \leqq 1$. ∎

5. Proof of Theorem 5

We prove part (b) first. Let $a > 0$, $b > 0$, θ_0, φ_0 be given, let $S = S(a, b, \theta_0, \theta_0)$, and assume that $\Sigma \cap S = \Sigma \cap \mathrm{bd}\, S \supseteq \{u, v\}$, where $u, v \in \mathbf{R}^2$ and $u \neq v (\mathrm{mod}\, \mathbf{Z}^2)$ (thus $a + b > 1$). We shall find u', v' on adjacent lines of Σ, that determine a line of negative slope, such that $\Sigma \cap S = \{u', v'\} + \mathbf{Z}^2$.

Define $\Sigma' = \Sigma - (\theta_0, \varphi_0)$, $S' = S(a, b, 0, 0)$ $(= S - (\theta_0, \varphi_0))$. We shall show that $E \cap \Sigma' \cap S' = E \cap \Sigma' \cap \mathrm{bd}\, S' = \{(\xi,\zeta), -(\xi,\zeta)\}$, where $0 < \xi < \frac{1}{2}$, $-\frac{1}{2} < \zeta < 0$, and the points (ξ,ζ) and $-(\xi,\zeta)$ lie on adjacent lines of Σ'. Putting $u' = (\theta_0, \varphi_0) + (\xi,\zeta)$, $v' = (\theta_0, \varphi_0) - (\xi,\zeta)$, we see that all the requirements are fulfilled.

By Lemma 3(b), $S' \cap E$ is strongly starshaped with respect to the origin. But $\Sigma' \cap \mathrm{int}\, S' = \emptyset$, so that $E \cap S'$ must be bounded by the two lines L_1 and L_2 of Σ' adjacent to the origin. S' is centrally symmetric and we claim that so is Σ'. Assume the contrary; then L_1, say, is nearer $(0,0)$ than L_2. By Lemma 8, Corollary 7(b) applies to L_1, and since $E \cap L_1 \cap \mathrm{int}\, S' = \emptyset$, we have $|E \cap L_1 \cap S'| \leqq 1$. By the central symmetry of S', $|E \cap S' \cap L_2| = 0$, and so $|E \cap S' \cap \Sigma'| \leqq 1$, a contradiction.

Thus Σ' and S' are symmetric, and therefore (w.l.o.g.) L_1 is the line $y = (q/p)x - 1/2p$ and L_2 is the line $y = (q/p)x + 1/2p$. Assuming that $E \cap S \cap L_1 = \{(\xi,\zeta)\}$, so that $E \cap S \cap L_2 = \{(-\xi, -\zeta)\}$, it remains to show that the points (ξ,ζ) and $-(\xi,\zeta)$ determine a line of negative slope. The symmetry of $E \cap S'$ (with respect to the axes) implies that $E \cap S'$ is bounded by the rhombus determined by the lines L_1, L_2 and their reflections $y = -(q/p)x + 1/2p$, $y = -(q/p)x - 1/2p$. The vertices of this rhombus are $(0, \pm 1/2p)$ and $(\pm 1/2q, 0)$. Since $q > p \geqq 1$, it follows that $(\xi,\zeta) \in \mathrm{int}\, E$ (unless $\xi = 0$ and $\zeta = \pm\frac{1}{2}$, in which case $(\xi,\zeta) = -(\xi,\zeta) (\mathrm{mod}\, \mathbf{Z}^2)$, i.e. $|(S \cap \Sigma)/\mathbf{Z}^2| \leqq 1$, contrary to our assumption). L_1

and L_2 are clearly tangent to bd S at the points (ξ, ζ) and $(-\xi, -\zeta)$, respectively. Thus the slope of bd S at the points $\pm(\xi, \zeta)$ is equal to the slope of Σ', which is $q/p > 0$. By Lemma 7(c), $\pm(\xi, \zeta)$ are in the interior of the 4th and 2nd quadrants, and thus determine a line of negative slope.

PROOF OF PART (a). We are given points $u = (\theta_1, \varphi_1)$, $v = (\theta_2, \varphi_2)$ on adjacent lines of Σ, such that $\theta_1 > \theta_2$ and $\varphi_1 < \varphi_2$, and we are looking for $a > 0$, $b > 0$, and θ_0, φ_0 such that

$$\Sigma \cap S(a, b, \theta_0, \varphi_0) = \Sigma \cap \operatorname{bd} S(a, b, \theta_0, \varphi_0) = \{(\theta_1, \varphi_1), (\theta_2, \varphi_2)\} + \mathbf{Z}^2.$$

Put $\theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$, $\varphi_0 = \frac{1}{2}(\varphi_1 + \varphi_2)$, $(\xi, \zeta) = (\theta_1, \varphi_1) - (\theta_0, \varphi_0)$ and define $\Sigma' = \Sigma - (\theta_0, \varphi_0)$. Note that $(\theta_2, \varphi_2) - (\theta_0, \varphi_0) = -(\xi, \zeta)$. If we find positive a, b such that for $S' = S(a, b, 0, 0)$, $E \cap S' \cap \Sigma' = E \cap \operatorname{bd} S' \cap \Sigma' = \{\pm(\xi, \zeta)\}$ then $S(a, b, \theta_0, \varphi_0)$ will satisfy our requirements.

We require that $\pm(\xi, \zeta) \in \operatorname{bd} S'$ so that a and b must satisfy:

$$a \cos 2\pi\xi + b \cos 2\pi\zeta = 1. \tag{5.1}$$

Also, because $E \cap S'$ is strongly starshaped (Lemma 3(b)) $E \cap S'$ should be contained in the closed strip between the two adjacent lines of Σ', L_1 through (ξ, ζ) and L_2 through $-(\xi, \zeta)$. Thus L_1 and L_2 must support $E \cap S'$ and so be tangent to bd S' at (ξ, ζ) and $-(\xi, \zeta)$, respectively. This implies:

$$pa \sin 2\pi\xi + qb \sin 2\pi\zeta = 0. \tag{5.2}$$

The solution to (5.1) and (5.2) is:

$$a = \frac{1}{\Delta} \cdot q \sin 2\pi\zeta, \qquad b = -\frac{1}{\Delta} \cdot p \sin 2\pi\xi,$$

where $\Delta = q \sin 2\pi\zeta \cos 2\pi\xi - p \cos 2\pi\zeta \sin 2\pi\xi$.

The point (ξ, ζ) lies on the line L_1: $y = qx/p - 1/2p$ and in the interior of the 4th quadrant. Thus $(\xi, \zeta) \in \operatorname{int} E$, or more precisely, $0 < \xi < 1/2q \leqq \frac{1}{4}$ and $-\frac{1}{2} \leqq -1/2p < \zeta < 0$. It follows that $q \sin 2\pi\zeta < 0$, and $-p \sin 2\pi\xi < 0$. To show that a and b exist and are positive we must show that $\Delta < 0$, or, equivalently, that $(q/p)\operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi\zeta > 0$. Indeed,

$$\frac{q}{p}\operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi\zeta > \operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi\zeta \qquad (\text{since } 0 < \xi < \tfrac{1}{4}).$$

Note that since $q > p \geqq 1$,

$$\zeta = \frac{q}{p}\xi - \frac{1}{2p} \geqq \xi - \frac{1}{2}.$$

Also, $\operatorname{ctg} 2\pi y$ is decreasing on $]-\frac{1}{2}, 0[$. Therefore,

$$\operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi\zeta \geqq \operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi(\xi - \tfrac{1}{2}) = \operatorname{ctg} 2\pi\xi - \operatorname{ctg} 2\pi\xi = 0.$$

Thus a and b are positive, and the lines L_1 and L_2 are tangent to $\operatorname{bd} S'$ at the points (ξ, ζ) and $-(\xi, \zeta)$, respectively, which lie in $\operatorname{int} E$. By Corollary 7(c), $E \cap S' \cap (L_1 \cup L_2) = \{\pm(\xi, \zeta)\}$. Since $E \cap S'$ is strongly starshaped (Lemma 3(b)), it follows that $E \cap S'$ lies entirely in the closed strip between L_1 and L_2, and thus

$$E \cap \operatorname{bd} S' \cap \Sigma' = E \cap S' \cap \Sigma' = E \cap S' \cap (L_1 \cup L_2) = \{\pm(\xi, \zeta)\},$$

as required. Thus the proof is complete.

Acknowledgement

The contents of this paper form a part of the author's M.Sc. thesis (The Hebrew University of Jerusalem, 1982). The author is pleased to acknowledge his indebtedness to his supervisor, Professor M. A. Perles, for his help and encouragement.

References

1. D. Gale, *Neighborly and cyclic polytopes*, Proc. Symp. Pure Math., 7 (Convexity), 1963, pp. 225–232.

2. B. Grünbaum, *Convex Polytopes*, Interscience, London/New York/Sydney, 1967.